Uncertainty Quantification in Rothermel's Model Using an Efficient Sampling Method

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Abstract—The purpose of the present work is to quantify parametric uncertainty in Rothermel's wildland fire spread model (implemented in software such as Behave-Plus3 and FARSITE), which is undoubtedly among the most widely used fire spread models in the United States. This model consists of a nonlinear system of equations that relates environmental variables (input parameter groups) such as fuel type, fuel moisture, terrain, and wind to describe the fire environment. This model predicts important fire quantities (output parameters) such as the head rate of spread, spread direction, effective wind speed, and fireline intensity. The proposed method, which we call sensitivity derivative enhanced sampling (SDES), exploits sensitivity derivative information to accelerate the convergence of the classical Monte Carlo method. Coupled with traditional variance reduction procedures, it offers up to two orders of magnitude acceleration in convergence, which implies that two orders of magnitude fewer samples are required for a given level of accuracy. Thus, it provides an efficient method to quantify the impact of input uncertainties on the output parameters.

Introduction

One of the primary goals of wildland fire management is to minimize the negative impact of fire on property and society through prevention and research. To meet this goal, fire researchers employ a variety of tools such as satellite imagery, experiments, fire danger indices, as well as mathematical models. A mathematical model typically consists of a set of nonlinear equations that describe the interaction of the various environmental variables. These equations can be used to predict valuable fire environment information such as the head rate of spread and spread direction.

Fire models typically fall into one or more of the following categories: physics-based, derived empirically, or constructed from statistical considerations. A fire model that is physics-based uses physical principles such as conservation of mass and energy to derive a formula for the rate of spread and other quantities of interest (see Weber 2001 and the references therein for an in-depth discussion). It is also possible to use a statistical description of test fires to predict fire behavior occurring under similar conditions. The McArthur models (McArthur 1966) used for grassland and forest fires in Australia are one such example. Finally, laboratory experiments can be performed to empirically determine quantities such as the propagating flux, which can, in turn, be used to obtain an expression for the rate of spread. Rothermel's model (Rothermel 1972), a fire spread model that spans the physical and empirical classes, is perhaps the best known model in the United States, and although more recent models include a wider range of fire phenomena, it is still in wide use today.

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Not all fire models, however, can be employed as efficient fire prediction tools. Some of the more complex models, which couple atmospheric and fire behavior effects, for instance, are currently too computationally expensive to serve as viable real-time prediction tools. These complex models, nevertheless, assist researchers in gaining a more profound understanding of fire behavior.

Rothermel's Fire Spread Model

Rothermel's wildland fire spread model was one of the first models to describe the fire environment through equations derived, for the most part, from thermodynamic principles. The fire environment describes the complex chemical and physical interaction of fuels, terrain, and weather. The term fire behavior is used to describe the physical characteristics of a fire such as its rate of spread, fireline intensity, flame length, and so forth. In North America, where forests and grasslands provide an abundant source of wildland fuel, wildland fires are of particular interest. Wildland fuels are fuels that consist primarily of vegetation, both live and dead, but may also include organic layers within the soil. A surface fire, the type of fire Rothermel's model was developed for, spreads through a layer of contiguous fuel extending from the ground up to approximately 2 m.

Rothermel's model groups input parameters into four main categories: fuel type, fuel moisture, topography, and wind. Some simplifying assumptions regarding the fuels are that for a small area and short-time periods, the fuel is taken to be homogeneous. The output variables we shall consider are the *rate of spread* (*ros* in m/s), the *direction of maximum spread* (*sdr* in °), and the *effective wind speed* (*efw* in m/s).

Parametric Uncertainty in Fire Spread Models

To properly use a fire model, it is essential to understand its limitations and scope of applicability. However, even when fire models are used adequately, discrepancies between the observed phenomena and model results are inevitable as models are derived under idealized conditions. Model errors can result from several factors including inadequate physical description, numerical errors, and parametric uncertainty (Walters and Huyse 2002). In this work we concentrate our efforts solely on those errors originating from parametric uncertainty. In order to quantify the impact of parametric uncertainty, it is important to describe the uncertainty mathematically. Because the value of an input parameter is seldom known exactly, a common approach (and the one we will pursue) is to assign it a mean value and an associated probability density function. The standard deviation can then be taken as a measure of the uncertainty in the parameter value. The impact of parametric uncertainty on the results can then be estimated using, for instance, a Monte Carlo simulation.

Sometimes the uncertainty associated with an input parameter is not only a consequence of the intrinsic complexity of the phenomena being modeled. Instead, uncertainties may be the inevitable byproducts of economic and efficiency constraints. For example, it may be expensive or intractable to measure fuel data directly as in the case of a large area.

On the other hand, it is not always necessary to concern ourselves with the uncertainty associated with every single parameter. A sensitivity analysis (Saltelli and others 2004) can help us identify the input parameters that have the greatest influence on output variables. Those parameters that have only a marginal impact on the quantities of interest can be assigned constant characteristic values to reduce computational demands.

Problem Formulation

We will focus on quantifying the impact and propagation of parametric uncertainty only on the following output variables: the *rate of spread* (*ros* in m/s), the *direction of maximum spread* (*sdr* in °), and the *effective wind speed* (*efw* in m/s). Using the same notation as in Bachmann (2001) (the full set of equations as well as their derivatives can also be found there), Rothermel's steady-state rate of spread is given by

$$ros = \frac{I_R \xi (1 + \Phi_w + \Phi_s)}{\rho_b \varepsilon Q_{ig}}, \tag{1}$$

where I_R is the reaction intensity, ξ is the propagating flux ratio, ρ_b is the ovendry bulk density, ε is the effective heating number, Q_{ig} is the heat of pre-ignition. Φ_w and Φ_s are the wind and slope correction factors, respectively. The output variables depend on the following parameters: fuel loading $w0_{d1}$, $w0_{d2}$, $w0_{d3}$, $w0_{lh}$, $w0_{lw}$, (in kg/m^2), surface-area-to-volume ratio sv_{d1}, sv_{d2}, sv_{d3}, sv_{lh}, sv_{lw}, (in 1/m), fuel moisture content m_{d1}, m_{d2}, m_{d3}, m_{lh}, m_{lw}, (in %), fuel bed depth d (in m), wind speed *wsp* (in m/s), wind direction θ (in °), aspect ratio *asp* (in °), and slope *slp* (in rad). The subscripts *d1, d2, d3, lh, lw,* denote the size classes traditionally used to categorize the different fuel moisture timelag classes (see Deeming and others 1978): dead fuel 0 – 0.6 cm, dead fuel 0.6 – 2.5 cm, dead fuel 2.5 – 7.5 cm, live herbaceous fuel, and live woody fuel, respectively. Throughout this paper we assume that those parameters that are not held constant follow a normal distribution with a given mean and standard deviation.

The primary focus of this work is to quantify the propagation and impact of parametric uncertainty via an efficient Monte Carlo method. The Monte Carlo convergence rate is accelerated through the use of a sensitivity derivative enhanced sampling method that exploits derivative information of the output functions with respect to the input parameters to make more judicious use of the samples generated in a simulation. We estimate the mean and standard deviation of the output variables using a traditional Monte Carlo method as well as with the sensitivity derivative enhanced sampling method (SDES). We compare the advantages of SDES over the Monte Carlo method via improvement ratios and timing performance. The distributions of output variables will also be generated. It should be noted that the computation of the required sensitivity derivatives accounts for only a fraction of the total cost of a simulation; these can be easily extracted using, for instance, an automatic differentiation package.

Numerical Method

To investigate the propagation and impact of input variable uncertainties, Bachmann and Allgower (2002) used a first-order Taylor method in place of a full-fledged Monte Carlo simulation to avoid the prohibitive computational expense incurred through a direct application of the classical Monte Carlo method. Indeed, because of its slow convergence rate and the costly generation of correlated input variables in the multivariate case, Monte Carlo methods are usually reserved to establish a reference against which other methods are

compared. However, by identifying and generating stochastic versions of only those parameters to which the output variables are most sensitive and at the same time improving the convergence rate of traditional Monte Carlo methods, it is possible to perform simulations utilizing the original model to capture the more intricate behavior that a low-order approximation might otherwise sacrifice. The modified Monte Carlo method we describe below is a step toward this goal.

Although Monte Carlo methods have long been popular because they are simple to implement and use the underlying model as a ``black box,'' their slow convergence rate often proves to be too inefficient especially when multiple simulations are required. A common approach to improve the convergence rate, which is known to be proportional to the variance of the objective function, is to reduce the variance via a suitable reformulation of the problem. Variants of this approach encompass a large class of methods collectively known as variance-reduction methods. The SDES method, which we describe below, is a variance-reduction method that has already been employed with success in fields such as optimal control (Cao and others 2003, 2004) and computational fluid dynamics (Mathelin and others 2004). In this section we review the theory underlying the method; our discussion closely follows Cao and others (2004, 2006). The methods we shall employ are described in their proper mathematical setting, but for the reader who is unfamiliar with some of the concepts discussed below, the textbooks by Ross (1997) and Shiryayev (1984) should elucidate some of the mathematical details that are omitted.

Monte Carlo Method

Suppose X is a random variable with finite expectation and let $p(x)(x \in \mathbb{R})$ be an associated probability density function (pdf). If $f : \mathbb{R} \to \mathbb{R}$ is a smooth function of X, we recall that the expectation $Ef(X)$ of f is defined by

$$Ef(X) := \int f(x)p(x)dx, \qquad (2)$$

where the integration is taken over the support of the pdf. The variance $Vf(X)$ is defined by

$$Vf(X) := E(f(X) - Ef(X))^2. \qquad (3)$$

For brevity, we will sometimes write μ_x and σ_x^2 for the expectation and variance of the random variable X, respectively.

In the classical Monte Carlo method, we estimate $Ef(X)$ by

$$Ef(X) \approx \frac{1}{N} \sum_{i=1}^{N} f(x_i). \qquad (4)$$

The N samples $x_1, \ldots, x_N$ are generated according to the probability density of X. The convergence of this estimate to $Ef(X)$ as $N \to \infty$ is guaranteed by the large number theorem (Shiryayev 1984; Ross 1997).

It is well-known that the approximation error made using (4) is proportional to $\frac{\sqrt{Vf(X)}}{\sqrt{N}}$. For computationally intensive problems this slow convergence might render the Monte Carlo approximation impractical. In the results section, the extent to which the sensitivity derivative Monte Carlo method alleviates this slow convergence will be shown.

Sensitivity Derivative Enhanced Sampling

Recently, Cao and others (2003, 2004) developed a variance-reduction method that exploits information regarding the sensitivity of f with respect to the random variable X (measured via derivatives of f with respect to X) to speed up the convergence of the Monte Carlo method. The result of their efforts was the sensitivity derivative enhanced sampling Monte Carlo method (SDES). The first-order SDES method is described below.

Given the first-order Taylor expansion of f about μ_x

$$J_1(x) := f(\mu_x) + f'(\mu_x)(x - \mu_x), \tag{5}$$

and using

$$\int p(x)dx = 1 \quad \text{and} \quad \int (x - \mu_x)p(x)dx = 0,$$

it is clear that

$$\int (f(x) - J_1(x))p(x)dx = \int f(x)p(x)dx - f(\mu_x).$$

Upon rearranging for $Ef(X)$, this suggests the sensitivity derivative Monte Carlo approximation of the expectation of f

$$Ef(X) \approx f(\mu_x) + \frac{1}{N}\sum_{i=1}^{N}(f(x_i) - J_1(x_i)). \tag{6}$$

The N samples are again generated according to the pdf of X.

The following inequalities illustrate the extent to which the variance of (6) is reduced. Let

$$m_1 = \max_{s \in \mathbb{R}}|f'(s)| \quad \text{and} \quad m_2 = \max_{s \in \mathbb{R}}|f''(s)|.$$

Then
$$Vf(X) \leq 2m_1^2 V(X)$$

$$V(f - J_1) \leq \frac{m_2^2}{2}(V(X)^2 + E((X - \mu_x)^4)). \tag{7}\tag{8}$$

Where (7) and (8) indicate that the SDES method is most efficient when $V(X)$ is small ($V(X) << 1$). See Cao and others (2006) for a rigorous proof of these results as well as a generalization to the nth-order SDES method.

SDES for Rothermel's Model

Although in this article we concentrate our efforts on Rothermel's model, we will state the mathematical model as a general nonlinear system of equations. The SDES method is applicable to any fire behavior model satisfying the appropriate smoothness assumptions.

Let the vector $X = (X_1, \ldots, X_m)$ represent the ensemble of input parameters that comprise the local fire environment and suppose $y = f(X)$ is a function of the random variable vector X. Here y may represent the effective wind speed efw, the maximum rate of spread ros, or the spread direction sdr. The vector X is composed of the fuel type, fuel moisture, terrain, and wind parameters. We shall denote the expectation of the parameter vector X by $\mu_x = (\mu_{x_1}, \ldots, \mu_{x_m})$ and the covariance of X by Σ. In this case the second-order SDES method is given by

$$Ef(X) \approx \frac{1}{N}\sum_{i=1}^{N}(f(x_i) - J_2(x_i)) + f(\mu_x) + \frac{1}{2}\text{trace}(\nabla^2 f(\mu_x)\Sigma), \tag{9}$$

where

$$J_2(x) = f(\mu_x) + \nabla f(\mu_x)(x - \mu_x) + \frac{1}{2}(x - \mu_x)^T \nabla^2 f(\mu_x)(x - \mu_x).$$

Here ∇f and $\nabla^2 f$ denote the gradient and Hessian of f, respectively.

To further improve the efficiency of the present sampling method, we couple SDES with stratified sampling. The standard stratified sampling technique is discussed in Cao and others (2004), for example.

Results and Discussion

To compare the efficiency of SDES with traditional Monte Carlo methods, we compute the rate of spread ros, effective wind speed efw, and spread direction sdr using two of the original fuel models found in Rothermel (1972): the short grass and chaparral fuel models. The main fuel parameters are summarized in tables 1 and 2. The following parameters are held constant throughout for both fuel models: the dead fuel moisture at 8 percent, the live fuel moisture at 150 percent, and the low heat content at 18622 kJ/kg. We shall also examine the additional speed-up obtained by coupling SDES with standard stratified sampling.

The uncertainty associated with an input parameter is described by assigning it a normal distribution with a typical mean (taken to be the value given in the original model) and a corresponding standard deviation (typically between 10 and 50 percent of the mean value). Two types of computations will be performed for each fuel model. First, we take the fuel bed depth d and the 1-h surface area/volume ratio sv_{d1} to be normally distributed random variables; all other parameters are fixed. Then, we include a random wind speed wsp and wind direction θ in addition to d and sv_{d1}.

To measure relative errors and improvement ratios of Monte Carlo approximations versus SDES, we use the L_2-norm (Euclidean norm). Recall that the L_2-norm is defined for a vector $x = (x_1, \ldots, x_n)$ as $\|x\|_2 = (x_1^2 + \cdots + x_n^2)^{1/2}$.

Table 1—Chaparral fuel model parameters.

Parameter	Symbol	μ	σ	Units
1-h fuel load	$w0_{d1}$	1.12	--	kg/m^2
10-h fuel load	$w0_{d2}$	0.90	--	kg/m^2
100-h fuel load	$w0_{d3}$	0.45	--	kg/m^2
Live herbaceous fuel load	$w0_{lh}$	0.0	--	kg/m^2
Live woody fuel load	$w0_{lw}$	1.12	--	kg/m^2
1-h surface area/vol. ratio	sv_{d1}	6562	740	m^2/m^3
Live herb surface area/vol. ratio	sv_{lh}	4921	--	m^2/m^3
Live woody surface area/vol. ratio	sv_{lw}	4921	--	m^2/m^3
Dead fuel moisture of extinction	mx	20	--	%
Fuel bed depth	d	1.83	0.3	m
Midflame windspeed	wsp	2.3	0.5	m/s
Direction of wind vector (from upslope)	θ	45	20	°

Table 2—Short grass fuel model parameters.

Parameter	Symbol	μ	σ	Units
1-h fuel load	$w0_{d1}$	0.17	--	kg/m^2
10-h fuel load	$w0_{d2}$	0.0	--	kg/m^2
100-h fuel load	$w0_{d3}$	0.0	--	kg/m^2
Live herbaceous fuel load	$w0_{lh}$	0.0	--	kg/m^2
Live woody fuel load	$w0_{lw}$	0.01	--	kg/m^2
1-h surface area/vol. ratio	sv_{d1}	11483	1150	m^2/m^3
Live herb surface area/vol. ratio	sv_{lh}	4921	--	m^2/m^3
Live woody surface area/vol. ratio	sv_{lw}	4921	--	m^2/m^3
Dead fuel moisture of extinction	mx	12	--	%
Fuel bed depth	d	0.30	0.05	m
Midflame windspeed	wsp	2.3	0.5	m/s
Direction of wind vector (from upslope)	θ	45	20	°

If $E^{MC} = (E_1^{MC}, \ldots, E_M^{MC})$ is a sequence of relative errors obtained from M Monte Carlo simulations, then a measure of the average error is given by

$$\left\| E^{MC} \right\|_2 = \left(\frac{1}{M} \sum_{i=1}^{M} [E_i^{MC}]^2 \right)^{1/2}.$$

The L_2 improvement ratio is then computed from

$$I_2 = \frac{\left\| E^{MC} \right\|_2}{\left\| E^{SDES} \right\|_2}.$$

Suppose, for example, that we use a Monte Carlo and a first-order SDES simulation to approximate the rate of spread using 5,000 samples. If the improvement ratio is $I_2 = 10$, then the average relative error obtained from SDES is 10 times smaller than that obtained from a Monte Carlo method using the same number of samples. Put another way, if we use SDES we need only 500 samples to achieve the same accuracy as the Monte Carlo method.

Table 3 shows that even with a simple first-order SDES method the convergence rate over a traditional Monte Carlo simulation can be as much as 20 times faster. It is important to note that SDES might require the computation of several derivatives of the objective function. In our computations, an automatic differentiation package (see Stamatiadis and others 2000) was used to find the relevant derivatives. Table 4 illustrates that even when we couple SDES with stratified sampling (denoted by SSD1, for a first-order SDES with stratified sampling), the extra computational expense incurred is marginal.

Table 3—SDES error improvement ratios for first-moment estimates of the rate of spread using $N = 512$ and $M = 100$ different sets of samples.

Fuel model	MC ratio (2vars)	MC ratio (4vars)	S = 4 ratio (2vars)	S = 2 ratio (4vars)
Chaparral	29.8	5.1	11.4	3.4
Short grass	24.2	4.3	10.0	3.1

Table 4—Timing results (in seconds). Average computational time comparison of Monte Carlo versus first-order SDES coupled with stratified sampling (four strata).

| N	(Chap) MC	(Chap) SSD1	(Shtgrs) MC	(Shtgrs) SSD1
32	4.297E–02	4.437E–02	4.063E–02	4.344E–02
64	8.188E–02	8.250E–02	7.922E–02	8.250E–02
128	1.617E–01	1.656E–01	1.567E–01	1.566E–01
256	3.231E–01	3.295E–01	3.113E–01	3.158E–01
512	6.423E–01	6.255E–01	6.216E–01	6.262E–01

Figure 1 illustrates that although the input parameters follow a normal distribution (as described in tables 1 and 2), this is not the case with the output functions $efw, ros,$ and sdr. This is to be expected as the output functions depend nonlinearly upon the random parameters. Using four random variables in the chaparral fuel model, the mean values are given by $efw_\mu = 2.41$ m/s, $ros_\mu = 0.353$ m/s, and $sdr_\mu = 41.3°$. The corresponding standard deviations are $efw_\sigma = 2.0325 \cdot 10^{-5}$ m/s, $ros_\sigma = 2.8921 \cdot 10^{-4}$ m/s, and $sdr_\sigma = 5.0674 \cdot 10^{-4}$. The standard deviation provides us with a measure of the uncertainty in the outputs.
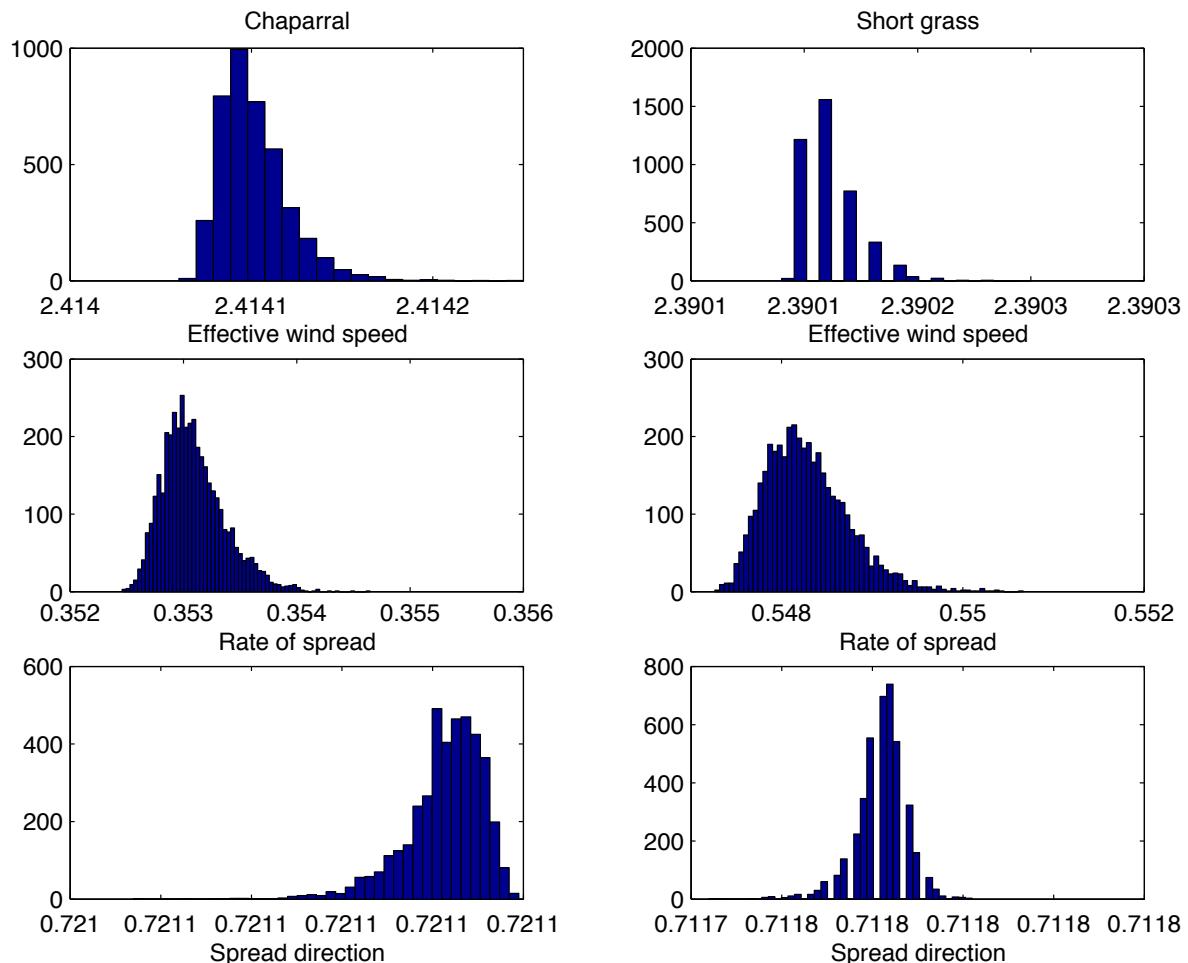

Figure 1—Output distributions of the effective wind speed, rate of spread, and spread direction using four random variables ($d, sv_{d1}, wsp,$ and θ). The mean values for the Chaparral fuel model are given by $efw_\mu = 2.41$ m/s, $ros_\mu = 0.35$ m/s, and $sdr_\mu = 41.3°$.

Figures 2 and 3 <u>show</u> that the relative errors decay at the expected rate proportional to $1/\sqrt{N}$, where N is the number of samples used. Throughout, we use $M = 100$ different sets of samples and then average the relative errors. We observe that Monte Carlo coupled with stratified sampling is as much as five times faster than the traditional Monte Carlo method. In all cases we observe that first-order SDES produces results that are up to two orders of magnitude more accurate (when coupled with stratified sampling) than plain Monte Carlo. With this convergence rate, it would take a Monte Carlo method as many as 100 times more samples to achieve comparable results.
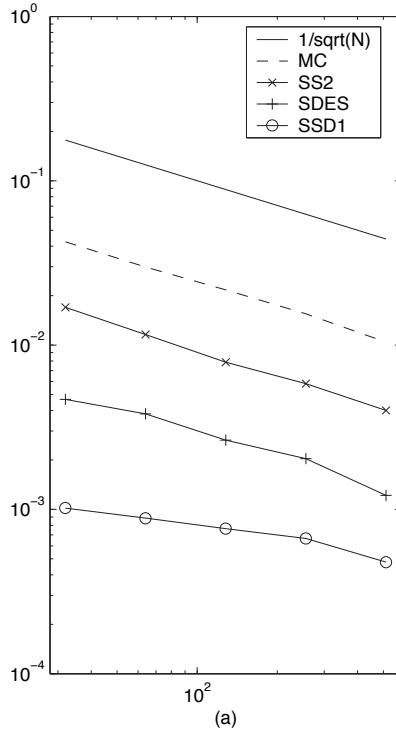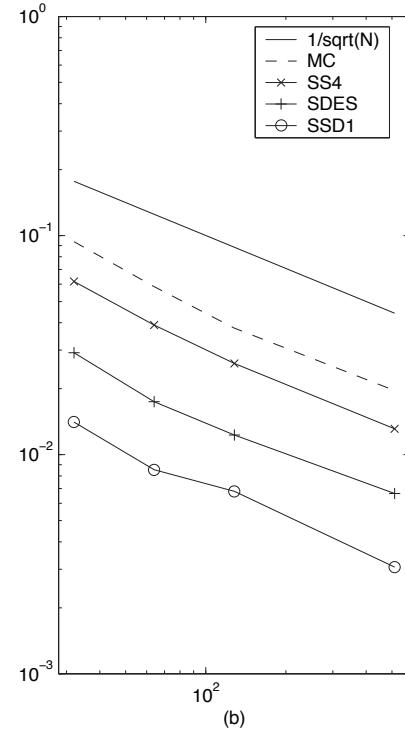
Figure 2—(Chaparral fuel model) Average relative errors in first moment estimates for the rate of spread *ros* using (a) two random variables $sv_{d1} \sim N(6562,740)$ and $d \sim N(1.83,0.3)$ and (b) four random variables $sv_{d1} \sim N(6562,740)$, $d \sim N(1.83,0.3)$, $wsp \sim N(2.3,0.5)$, and $\theta \sim N(45,20)$.
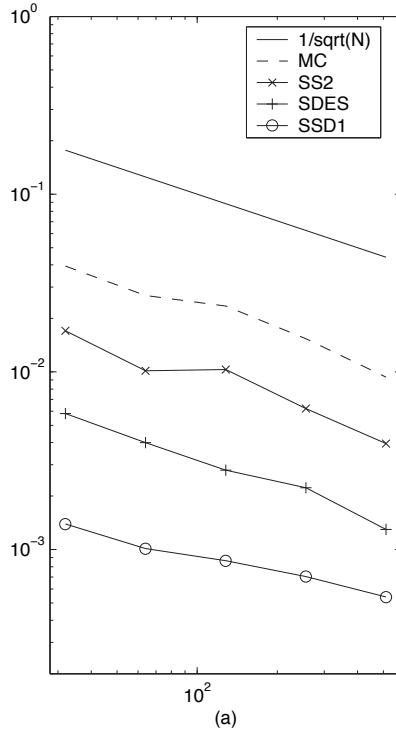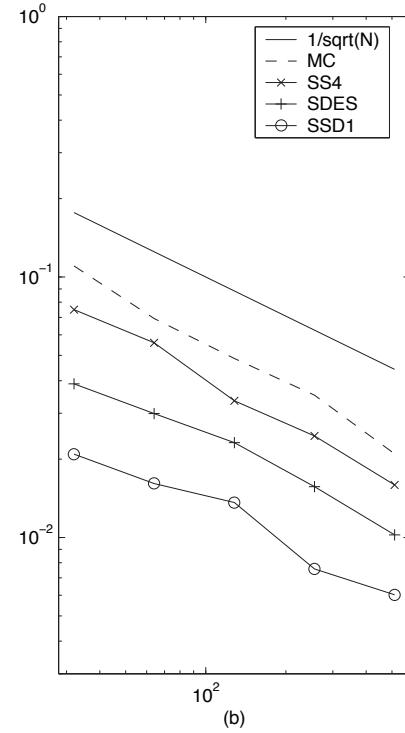
Figure 3—(Short grass fuel model) Average relative errors in first moment estimates for the rate of spread *ros* using (a) two random variables $sv_{d1} \sim N(11483,1150)$ and $d \sim N(0.30,0.05)$ and (b) four random variables $sv_{d1} \sim N(11483,1150)$, $d \sim N(0.30,0.05)$, $wsp \sim N(2.3,0.5)$, and $\theta \sim N(45,20)$.

Conclusions

Although the simplicity of the Monte Carlo method makes it an attractive method to use in simulations that use fire spread models such as Rothermel's model, its slow convergence can render its application infeasible especially in time-sensitive situations such as in the prediction of an ongoing fire. In this work we have demonstrated that the sensitivity derivative enhanced sampling method and its variants provide fire researchers an economic alternative to traditional Monte Carlo methods in quantifying parametric uncertainty. Quantifying the impact of parametric uncertainty is of utmost importance since key input parameters used by fire spread models are seldom known exactly. The speed-up of up to two orders of magnitude in the SDES gives fire managers the ability to effectively and efficiently run simulations in real-time using only minimal computational resources.

The results indicate that coupling SDES with stratified sampling can further accelerate the convergence rate. This suggests that coupling SDES with more sophisticated sampling techniques such as Latin hypercube sampling or orthogonal sampling, while not as easily implemented as stratified sampling, might improve the convergence rate even further. We will explore these possible enhancements in future investigations.

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